A model for time-dependent cosmological constant

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Abstract

We present a model for a spacetime dependent cosmological *constant*. We make a realization of this model based on a possible quantum aspects of the initial stage of the universe and relate the cosmological constant with the chiral anomaly.

I. INTRODUCTION

As Weinberg states very convincingly: "We want to explain why the effective cosmological *constant* is small now, *not* why it was always small." In other words, we are looking for a dynamical treatment of Λ.

In the last years a proliferous series of papers dealing with a time dependent cosmological constant appeared[1]. The main reason for such interest is related to the possibility to treat the vacuum dynamically, as it was required for many inflationary scenarios. It is not surprising that it has revealed a simpler task to provide a model for a varying Λ than it was for the case of a constant one[2]. A certain number of different models appeared. Almost all of them deal with scalar-tensor theory of gravity. We can quote, for instance [1] and references therein. We also mention that since the paper by Özer-Taha [2], many authors have been using this spacetime dependence of Λ with the purpose of eliminating some difficulties from the standard cosmological model, mainly those related to the flatness problem.

In the present paper we propose a distinct model for a varying Λ, based on a effective Lagrangian depending on a gauge invariant quantity, that is constructed in terms of a field strength and its dual. After that, we try to associate this mechanism to a possible quantum behavior of the initial stage of the universe. Let us remark that nowadays we are almost all convinced that the evolution of the Universe has been proceeded most classically. Some quantum effects might take place at black holes [3], but there is no evidence of such phenomena till now. It is then widely believed that cosmology is essentially a classical subject. However, in the beginning stage of the expansion of the Universe evolution, where the matter content was formed just by elementary particles, probably quarks and leptons, immerse in a very hot environment, it was much more quantum. We are going to show that the chiral anomaly Lagrangian can be conveniently adapted in order to generate a spacetime dependent cosmological *constant*. This procedure would also naturally explain why the cosmological constant is small now. Its most relevant value would be related with the period that the universe evolved quantically.

The problem is that we do not know yet a consistent and complete quantization of gravitation. A way of circumventing this problem is by means of a semiclassical treatment, where just matter fields are quantized and propagate in a classical curved background [4]. For our particular purposes in the present work, we consider that this background also contains classical gauge fields.

Our paper is organized as follow. In Sec. II we present the general model to obtain a spacetime dependent cosmological *constant*. In Sec. III we make a realization of this model by means of a semiclassical treatment of matter fields interacting with a classical curved background also containing gauge fields. We left Sec. IV for concluding remarks and introduce an Appendix to give some details of the calculation of the chiral anomaly in curved spacetime by using the zeta function regularization.

II. THE MODEL

We introduce in this section a general model that can generate a time dependent cosmological *constant*. Consider a Lagrangian L given by

$$L = L_{mat} + Y(\mathcal{G}) \qquad (2.1)$$

where Y is a nonlinear function of an invariant quantity $\mathcal{G}$ which is constructed in terms of a gauge field strength $G_{\mu\nu}{}^a$ of a given group G and its dual $G_{\mu\nu}^{*}{}^a = \frac{1}{2}\eta_{\mu\nu\rho\lambda}G^{\rho\lambda a}$. We are using the following definition for $\eta_{\mu\nu\rho\lambda}$

$$\eta_{\mu\nu\rho\lambda} = \sqrt{-g}\,\epsilon_{\mu\nu\rho\lambda} \tag{2.2}$$

Consequently,

$$\eta^{\mu\nu\rho\lambda} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\rho\lambda} \tag{2.3}$$

where $\epsilon_{\mu\nu\rho\lambda}$ and $\epsilon^{\mu\nu\rho\lambda}$ are the usual Levi-Civita tensor densities ($\epsilon_{0123} = 1$) and g is the determinant of the metric tensor.

For our purposes here it is not necessary to specify the gauge group, and from now on we just report to the Abelian case for simplicity. So

$$\mathcal{G} \equiv G_{\mu\nu}^{*}\,G^{\mu\nu} \tag{2.4}$$

In order to obtain from this form of L the expression of $T_{\mu\nu}$ we have to vary the Lagrangian with respect to $g^{\mu\nu}$. The important property that we need is

$$\delta\mathcal{G} = \frac{1}{2}\,\mathcal{G}\,g_{\mu\nu}\,\delta g^{\mu\nu} \tag{2.5}$$

From this expression and using the definition of the energy-momentum tensor in terms of variation of the background metric provided by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta L\sqrt{-g}}{\delta g^{\mu\nu}} \tag{2.6}$$

we obtain

$$T_{\mu\nu} = T_{\mu\nu}^{(mat)} + \Lambda\,g_{\mu\nu} \tag{2.7}$$

in which

$$\Lambda = \frac{dY}{d\mathcal{G}}\,\mathcal{G} - Y \tag{2.8}$$

From this expression it follows that the effective Lagrangian contributes to the energy-momentum tensor with a term that is proportional to the metric tensor of the background geometry. This is interpreted, in the Einstein General Relativity theory as a spacetime dependent cosmological *constant*. We observe that if Y is linear in $\mathcal{G}$, there is no contribution for Λ.

It is opportune to mention that besides the ordinary interaction of matter with the gauge field A, through a conserving current J_A^μ, the field can interact with axionic matter a via a Lagrangian that depends on the product $a\,\mathcal{G}$ [5]

$$L_{int} = L(a\,\mathcal{G}) \tag{2.9}$$

The presence of this interaction will not affect the general lines of the mechanism to obtain the spacetime dependent cosmological *constant* but, of course, it will modify its value [3].

III. A QUANTUM ORIGIN FOR Λ

In this section we associate the Lagrangian Y to a possible quantum scenario of the beginning universe. This would lead to a natural time dependence of the cosmological *constant*, in a sense that it would be relevant just in the period of this quantum scenario.

Let us then consider an idealized situation with the universe being filled with quantum (fermionic) matter fields ψ interacting with some external gauge fields A_μ and also with an strong gravitational background[4]. We have the following action

$$S = i\int d^4x\sqrt{-g}\,\bar{\psi}\gamma^\mu\big(\nabla_\mu - ie\,A_\mu\big)\psi \tag{3.1}$$

where γ^μ are the curved-space Dirac matrices, depending on the spacetime, that are related to a local flat-space by

$$\gamma^\mu(x) = e_a^\mu(x)\gamma^a \tag{3.2}$$

e_a^μ are the tetrad fields and satisfy the standard relations

$$\begin{aligned}
e_{\mu a}e_\nu^a &= g_{\mu\nu}\\
e_{\mu a}e^{\nu a} &= \delta_\mu^\nu\\
e_{\mu a}e^{\mu b} &= \delta_a^b\\
e_{\mu a}e_b^\mu &= \eta_{ab}
\end{aligned} \tag{3.3}$$

Consequently,

$$\begin{aligned}
g &= \det\left(e_\mu^a\right)\det\left(e_\nu^b\right)\det\left(\eta_{ab}\right)\\
&= -\left(\det e\right)^2
\end{aligned} \tag{3.4}$$

The covariant derivative ∇_μ acting on the spinorial field ψ means

$$\nabla_\mu\psi = \left(\partial_\mu + \Gamma_\mu\right)\psi \tag{3.5}$$

where $\Gamma_\mu(x) = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab}$ is the spin connection. Here, Σ^{ab} are flat-space time matrices defined by $\Sigma^{ab} = \frac{1}{4}[\gamma^a,\gamma^b]$. The quantities $\omega_{\mu ab}$ can be written in terms of the tetrad fields by imposing that the covariant derivative of the connection is zero. The result is

$$\omega_{\mu ab} = \frac{1}{2}\Big[e_a^\nu\left(\partial_\mu e_{\nu b} - \partial_\nu e_{\mu b} + e_\mu^c e_b^\rho\partial_\rho e_{\nu c}\right) - (a\leftrightarrow b)\Big] \tag{3.6}$$

We have not considered the mass of particles by virtue of the high energies evolved in the initial stage of the universe. Since there is no mass, the action (3.1), besides gauge invariance, is also invariant by chiral gauge transformation, namely

$$\delta\psi(x) = -i\gamma_5\xi(x)\psi(x) \tag{3.7}$$

The $\gamma_5(x)$ matrix in curved space is the same as the usual flat γ_5, i.e.

$$\begin{aligned}
\gamma_5(x) &= i\sqrt{-g}\,\gamma^0(x)\gamma^1(x)\gamma^2(x)\gamma^3(x) \\
&= \frac{i}{4!}\sqrt{-g}\,\epsilon_{\mu\nu\rho\lambda}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda \\
&= \frac{i}{4!}\sqrt{-g}\,\epsilon_{\mu\nu\rho\lambda}e_a^\mu e_b^\nu e_c^\rho e_d^\lambda\gamma^a\gamma^b\gamma^c\gamma^d \\
&= \frac{i}{4!}\sqrt{-g}\,\epsilon_{abcd}\det(e_a^\mu)\gamma^a\gamma^b\gamma^c\gamma^d \\
&= \gamma_5 \tag{3.8}
\end{aligned}$$

The chiral invariance is not maintained in the quantum counterpart, what leads to an effective action given by (See Appendix A)

$$\begin{aligned}
S_{eff} &= S - \frac{ie^3}{16\pi^2}\int d^4x\,\xi(x)\,\epsilon^{\mu\nu\rho\lambda}G_{\mu\nu}G_{\rho\lambda} \\
&= S + \frac{ie^3}{16\pi^2}\int d^4x\sqrt{-g}\,\xi(x)\,\eta^{\mu\nu\rho\lambda}G_{\mu\nu}G_{\rho\lambda} \tag{3.9}
\end{aligned}$$

Even though the above equation contains a term that could be identified with $\mathcal{G}$ it would not generate a spacetime dependent cosmological *constant* because it is linear in $\mathcal{G}$. In order to associate this axial anomaly with the origin of $\Lambda(x)$, we redefine the gauge field A_μ like

$$A_\mu = B_\mu\,\phi \tag{3.10}$$

where ϕ is considered to be some gauge invariant quantity. Consequently the gauge transformation of B_μ reads

$$\delta B_\mu = \frac{\partial_\mu\xi}{\phi} \tag{3.11}$$

Let us replace A_μ by $B_\mu\phi$ into the expression (3.9).

$$\begin{aligned}
S_{eff} &= S + \frac{ie^3}{4\pi^2}\int d^4x\,\sqrt{-g}\,\xi(x)\,\eta^{\mu\nu\rho\lambda}\partial_\mu(B_\nu\phi)\partial_\rho(B_\lambda\phi) \\
&= S + \frac{ie^3}{4\pi^2}\int d^4x\,\sqrt{-g}\,\xi(x)\,\eta^{\mu\nu\rho\lambda}(\partial_\mu B_\nu\phi + B_\nu\partial_\mu\phi)(\partial_\rho B_\lambda\phi + B_\lambda\partial_\rho\phi) \\
&= S + \frac{ie^3}{4\pi^2}\int d^4x\,\sqrt{-g}\,\xi(x)\,\eta^{\mu\nu\rho\lambda}(\partial_\mu B_\nu\partial_\rho B_\lambda\phi^2 + 2\partial_\mu B_\nu B_\lambda\phi\partial_\rho\phi) \\
&= S - \frac{ie^3}{4\pi^2}\int d^4x\,\sqrt{-g}\,\eta^{\mu\nu\rho\lambda}\partial_\mu B_\nu B_\lambda\phi^2\partial_\rho\xi \tag{3.12}
\end{aligned}$$

where we have disregarded total derivative terms inside the action. If one identifies ϕ with $\mathcal{G}$, what is in agreement with the gauge invariance condition for ϕ, we observe that the second term of (3.12) will be quadratic in $\mathcal{G}$. However, there is another part that is not topological invariant. Let us convenient rewrite the expression (3.12) as

$$S_{eff} = S + \int d^4x\,\sqrt{-g}\,\mathcal{K}\,\mathcal{G}^2 \tag{3.13}$$

where $\mathcal{K} = -\frac{ie^3}{4\pi^2}\eta^{\mu\nu\rho\lambda}\partial_\mu B_\nu B_\lambda\partial_\rho\xi$. Similarly to the variation of $\mathcal{G}$ with respect to the metric tensor, we also have for $\mathcal{K}$

$$\delta\mathcal{K} = \frac{1}{2}\,\mathcal{K}\,g_{\mu\nu}\delta g^{\mu\nu} \tag{3.14}$$

Even though the corresponding Lagrangian of Eq. (3.13) is slightly different of the term Y of the Lagrangian introduced in the initial model, we can show that this term also leads to a cosmological *constant* Λ given by

$$\Lambda = 2\mathcal{K}\mathcal{G}^2 \tag{3.15}$$

which shows that the chiral anomaly can be associated to a origin for the time-dependent cosmological *constant*.

It is opportune to remark that the above mechanism which generates the cosmological term Λ also requires a modification on the matter Lagrangian such that the total energy momentum tensor $T_{\mu\nu} = T_{\mu\nu}^{mat} + \Lambda g_{\mu\nu}$ is conserved.

IV. CONCLUSION

In the first part of this work we have presented a general model to obtain a spacetime dependent cosmological *constant*. After that, we have shown that this model can be realized by means of a semiclassical treatment of an idealized situation related to the primordial universe, where we consider it was filled with quantum fermionic

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matter interacting with a external gauge field and with a strong classical gravitational background. Our main purpose in associating this model to a possible quantum scenario of the primordial universe is that the so obtained cosmological *constant* would have a significant value just while the universe evolved quantically. This naturally explains why the cosmological *constant* is small nowadays, when the expansion of the universe is widely accepted to be classical.

The next natural step of this research is to consider this idea in some cosmological model in order to see how the cosmological term that comes from Eq. (3.12) will modify the dynamics of the Einstein equation. This is presently under study and possible results shall be reported elsewhere [10].

APPENDIX: APPENDIX A

In this Appendix, we briefly discuss the obtainment of the chiral anomaly by using the Lagrangian path integral formalism and the zeta function regularization, that are both consistent with the curved space formulation.

Let us write down the vacuum functional corresponding to the action S given by (3.1), i.e.

$$Z = \int [d\bar{\psi}][d\psi]\, e^{iS} \qquad (A.1)$$

The functional integrations are just over $\bar{\psi}$ and ψ because just the fermionic fields are considered to be quantum. They evolve in a curved classical background that also contains gauge fields.

We expand ψ and $\bar{\psi}$ in terms of a complete set of orthonormal eigenfunctions $\{\Phi_m\}$ of the operator $\not{D} = \gamma^\mu(\nabla_\mu - ieA_\mu)$, i.e.

$$\not{D}\Phi_m = \gamma^\mu(\nabla_\mu - ieA_\mu)\,\Phi_m = \lambda_m \Phi_m \qquad (A.2)$$

Since the set is complete and orthonormal we have

$$\int d^4x\, \sqrt{-g}\, \Phi_m^\dagger(x)\Phi_n(x) = \delta_{mn} \qquad (A.3)$$

$$\sum_m \Phi_m(x)\Phi_m^\dagger(y) = \frac{\delta(x-y)}{\sqrt{-g}} \qquad (A.4)$$

This allows expansions of the fermion fields $\psi(x)$ and $\bar{\psi}(x)$ as

$$\psi(x) = \sum_m a_m\, \Phi_m(x)$$

$$\bar{\psi}(x) = \sum_m \Phi_m^\dagger(x)\, b_m \qquad (A.5)$$

so that

$$[d\bar{\psi}][d\psi] = \prod_m db_m da_m \qquad (A.6)$$

a_m and b_m are elements of a Grassmannian algebra.

Considering the infinitesimal chiral transformations given by (3.7), we have

$$\psi'(x) = [1 - ie\gamma_5\xi(x)]\,\psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x)\,[1 - ie\gamma_5\xi(x)] \qquad (A.7)$$

and also expanding ψ' and $\bar{\psi}'$ in terms of the complete set $\{\Phi_m\}$, we obtain

$$\psi'(x) = \sum_m a_m'\, \Phi_m(x)$$

$$\bar{\psi}'(x) = \sum_m \Phi_m^\dagger(x)\, b_m' \qquad (A.8)$$

The combination of (A.5), (A.7), and (A.8) leads to

$$a_m' = \sum_n C_{mn}\, a_n$$

$$b_m' = \sum_n C_{mn}\, b_n \qquad (A.9)$$

where

$$C_{mn} = \delta_{mn} - ie \int d^4x\, \sqrt{-g}\, \xi(x)\, \Phi_m(x)\gamma_5\Phi_n(x) \quad (A.10)$$

In the case of Grassmann variables, we have

$$\prod_m da_m = \det(C_{mn}) \prod_m da_m'$$

$$\prod_m db_m = \det(C_{mn}) \prod_m db_m' \qquad (A.11)$$

Consequently

$$[d\bar{\psi}][d\psi] = \Big(\det(C_{mn})\Big)^2 [d\bar{\psi}'][d\psi'] \qquad (A.12)$$

Considering the expression for the matrix C_{mn} given by (A.10), we have

$$\det(C_{mn}) = \det\Big(\delta_{mn} - ie \int d^4x\, \sqrt{-g}\, \xi(x)\, \Phi_m^\dagger(x)\gamma_5\Phi_n(x)\Big)$$

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$$= \exp \operatorname{tr} \ln\left(\delta_{mn} - ie \int d^4x \sqrt{-g}\,\xi(x)\,\Phi_m^\dagger(x)\gamma_5\Phi_n(x)\right)$$

$$\simeq \exp \operatorname{tr}\left(-ie \int d^4x \sqrt{-g}\,\xi(x)\,\Phi_m^\dagger(x)\gamma_5\Phi_n(x)\right)$$

$$= \exp\left(-ie \int d^4x \sqrt{-g}\,\xi(x) \sum_m \Phi_m^\dagger(x)\gamma_5\Phi_n(x)\right) \qquad (A.13)$$

From (A.4) we see that the sum which appears in the expression above is badly divergent. We are going to regularize it by using the zeta function technique in curved spacetime [6]. The main result of this procedure can be resumed in the expression [7]

$$\sum_m \Phi_m^\dagger(x)\gamma_5\Phi_m(x) = \lim_{s\to 0} \operatorname{tr}[\gamma_5\zeta(x,s)] \qquad (A.14)$$

where $\zeta(x,s)$ is the generalized zeta function, which is related to the usual zeta function $\zeta(x)$ by

$$\zeta(s) = \operatorname{tr} \int d^4x \sqrt{-g}\,\zeta(x,s) \qquad (A.15)$$

We can obtain informations of the behavior of the generalized zeta function by means of the heat equation and one can show that the generalized zeta function can be associated to powers of s with coefficients depending on x, denoted by $[a_n(x)]$. In the particular case of the operator $\not{D}^2$ we obtain the general result

$$\lim_{s\to 0} \zeta(x,s) = \begin{cases} \frac{\sqrt{-g}}{(4\pi)^{d/2}}\,[a_{d/2}] & d \text{ even} \\ 0 & d \text{ odd} \end{cases} \qquad (A.16)$$

where d is the spacetime dimension. So, for $d=4$, $\zeta(x,0)$ is just related to $[a_2]$. These coefficients are already evaluated in literature for some specific operators. For the case of the operator $\not{D}^2$, we have [7]

$$[a_2] = -\frac{1}{120}\,R_{;\mu}{}^\mu + \frac{1}{288}\,R^2 - \frac{1}{180}\,R_{\mu\nu}R^{\mu\nu} + \frac{1}{180}\,R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$$
$$+ \frac{1}{24}\,[\gamma^\mu,\gamma^\nu]G_{\mu\nu;\rho}{}^\rho + \frac{1}{24}\,R\,[\gamma^\mu,\gamma^\nu]G_{\mu\nu} + \frac{1}{32}\,([\gamma^\mu,\gamma^\nu]G_{\mu\nu})^2 + \frac{1}{12}G_{\mu\nu}G^{\mu\nu} \qquad (3.17)$$

From (A.14) we notice that the regularization of the sum $\sum_m \Phi_m^\dagger\gamma_5\Phi_m$ contains γ_5 times $\zeta(x,0)$. Hence, just the term $\frac{1}{32}([\gamma^\mu,\gamma^\nu]G_{\mu\nu})^2$ will contributes for the sum, which leads to the anomaly term given by (3.9).

the same presented here.

[4]   The interaction of the matter with the G-field is given through a current $J_\mu{}^A$ but since it is conserved it does not contribute to the total $T_{\mu\nu}$.

[1] L.M.Diaz-Rivera and L.O.Pimentel, Phys. Rev. **D60** (1999) 123501.

[2] M. Özer and M.O. Taha, Nucl. Phys. **B287** (1987) 776.

[3] S. Hawking, Nature **248** (1974) 30.

[4] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge (1982).

[5] See for example E.W. Kolb and M.S. Tuner, *The early universe*, Addison-Wesley (1990) and references therein.

[6] See for example S.W. Hawking, Commun. Math. Phys. **55** (1977) 133 and references therein.

[7] See for example B.S. DeWitt, *The spacetime approach to quantum field theory*, Proceedings of *Relativity, groupos and topology II* (Les Houches, 1983) Editors: B.S. DeWitt and R. Stora.

[8] See J.M.Overduin and F.I.Cooperstock Phys. Rev. **D58** (1998) 043506.

[9] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1.

[10] M. Novello, J. Barcelos-Neto and J.M. Salim, work in progress.

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[1]   See [8] for a review and for a large list of references on this subject.

[2]   See [9] for a description of the different analysis of a constant Λ.

[3]   We are going to see in the next section that the realization of this model is achieved by a slightly more complicate quantity than $Y(\mathcal{G})$, but the general lines are precisely